Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **94**, 1–21; doi: 10.14232/ejqtde.2016.1.94 http://www.math.u-szeged.hu/ejqtde/

A variational property on the evolutionary bifurcation curves for the one-dimensional perturbed Gelfand problem from combustion theory

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Received 3 May 2016, appeared 13 October 2016
Communicated by Paul Eloe

Abstract. We study a variational property on the evolutionary bifurcation curves for the one-dimensional perturbed Gelfand problem from combustion theory

$$\begin{cases} u''(x) + \lambda \exp\left(\frac{au}{a+u}\right) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where $\lambda > 0$ is the Frank–Kamenetskii parameter or ignition parameter, $a > 0$ is the activation energy parameter, and u is the dimensionless temperature.

Keywords: positive solution, exact multiplicity, Turning point, S-shaped bifurcation curve.

2010 Mathematics Subject Classification: 34B18, 74G35.

1 Introduction and the main result

In this paper we mainly study a variational property on the evolutionary bifurcation curves of positive solutions for the two-point boundary value problem

$$\begin{cases} u''(x) + \lambda \exp\left(\frac{au}{a+u}\right) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{1.1}$$

which is the one-dimensional case of a problem arising in the study of standard models of ignition in a context of thermal combustion, cf. [1,14]. In (1.1), $\lambda > 0$ is the Frank–Kamenetskii parameter or ignition parameter, $a > 0$ is the activation energy parameter, u is the dimensionless temperature of the medium, and the reaction term

$$f(u) \equiv \exp\left(\frac{au}{a+u}\right)$$

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is the temperature dependence obeying the simple Arrhenius reaction-rate law in irreversible chemical reaction kinetics, see, e.g. Boddington et al. [2]. Notice that, substituting $a = 1/\varepsilon$ (ε is the reciprocal activation energy parameter) into (1.1), we obviously obtain

$$\begin{cases} u''(x) + \lambda \exp\left(\frac{u}{1+\varepsilon u}\right) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0. \end{cases} \tag{1.2}$$

This problem (1.2) is the famous one-dimensional perturbed Gelfand problem, cf. [1,3,5,10, 11,13].

For any $a > 0$, on the $(\lambda, \|u\|_\infty)$-plane, we study the shape and structure of bifurcation curves S_a of positive solutions of (1.1), defined by

$$S_a \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\}.$$

We say that, on the $(\lambda, \|u_\lambda\|_\infty)$-plane, the bifurcation curve S_a is S-shaped if S_a has *exactly two* turning points at some points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ where $\lambda_* < \lambda^*$ are two positive numbers such that

(i) $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$,

(ii) at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the bifurcation curve S_a turns to the left,

(iii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ the bifurcation curve S_a turns to the right.

See Figure 1.1 (i). In that case for S-shaped bifurcation curve S_a for thermal combustion problem (1.1), the two critical values λ^* and λ_* correspond to ignition limit and extinction limit respectively. The upper branch of S_a is then known as the explosion branch, and the lower branch the quenching branch. See [9, p. 374].

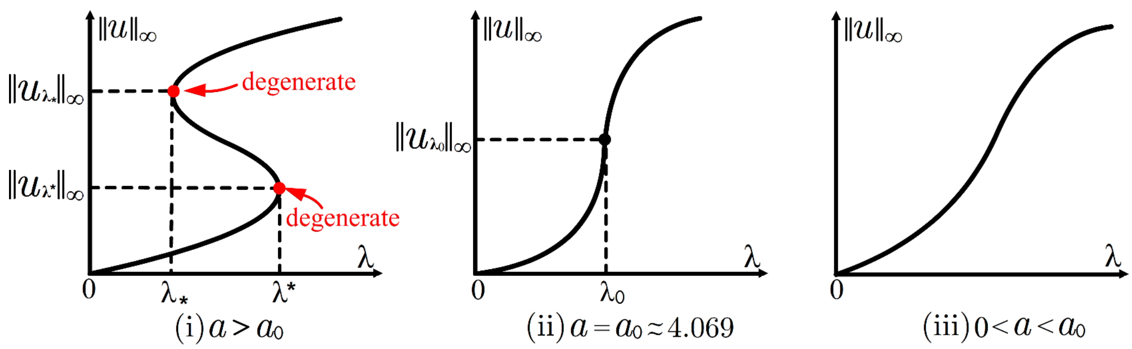

Figure 1.1: The global bifurcation of bifurcation curves S_a for $a > 0$.

Huang and Wang [6, Theorem 4] very recently studied global bifurcation of bifurcation curves S_a in the following theorem.

Theorem 1.1 (See Figure 1.1). *Consider* (1.1) *with varying $a > 0$. Then there exists a critical value $a_0 \approx 4.069$ such that the following assertions (i)–(iii) hold:*

(i) (See Figure 1.1 (i).) For $a > a_0$, the bifurcation curve S_a is S-shaped on the $(\lambda, \|u\|_\infty)$-plane. Let $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ be exactly two turning points of the bifurcation curve S_a satisfying $\lambda_* < \lambda^*$ and $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$. Then u_{λ_*} and u_{λ^*} are only two degenerate positive solutions of (1.1).

(ii) (*See Figure 1.1(ii).*) *For $a = a_0$, the bifurcation curve S_{a_0} is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. Moreover, (1.1) has exactly one (cusp type) degenerate positive solution u_{λ_0}.*

(iii) (*See Figure 1.1(iii).*) *For $0 < a < a_0$, the bifurcation curve S_a is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. Moreover, all positive solutions u_λ of (1.1) are nondegenerate.*

Furthermore, Hung and Wang [8] proved that there exists a positive number $a^*(\approx 4.166) > a_0$ such that

$$p_1(a) < \|u_{\lambda^*}\|_\infty < \gamma(a) < p_2(a) < \|u_{\lambda_*}\|_\infty \quad \text{for } a \geq a^*, \tag{1.3}$$

where

$$\gamma(a) \equiv \frac{a(a-2)}{2}, \quad p_1(a) \equiv \frac{a\,(a-2) - a\sqrt{a(a-4)}}{2}, \quad p_2(a) \equiv \frac{a\,(a-2) + a\sqrt{a(a-4)}}{2}. \tag{1.4}$$

Clearly, $p_1(a) < \gamma(a) < p_2(a)$ for $a > 4$. In addition, for $a > 4$, we note that $(\gamma(a), f(\gamma(a)))$ is the unique inflection point of $f(u)$ on $(0, \infty)$, and $p_1(a)$ and $p_2(a)$ are two positive zeros of

$$f(u) - uf'(u) = \frac{\left[u^2 - a(a-2)u + a^2\right]}{(a+u)^2} \exp\left(\frac{au}{a+u}\right), \tag{1.5}$$

which is the y-intercept of the tangent line to the graph of f at the point $(u, f(u))$. In this paper, we continue our work [6] and extend the result of (1.3). The following Theorem 1.2 is our main result, in which we show the variation of the values of $\|u_{\lambda^*}\|_\infty$ and $\|u_{\lambda_*}\|_\infty$ with varying parameter $a > a_0$, where $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ are defined in Theorem 1.1.

Theorem 1.2 (See Figures 1.1(i) and 1.2). *Consider (1.1) with varying $a > a_0$. Let $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ be two turning points of the bifurcation curve S_a satisfying $\lambda_* < \lambda^*$ and $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$. Then there exist two positive numbers $\hat{a} \approx 4.088$, $\check{a} \approx 4.077$ satisfying $a^* > \hat{a} > \check{a} > a_0$ such that:*

$$(1 <)\ p_1(a) < \|u_{\lambda^*}\|_\infty < \gamma(a) < p_2(a) < \|u_{\lambda_*}\|_\infty \quad \text{for } a > \hat{a}, \tag{1.6}$$

$$\gamma(\hat{a}) = \|u_{\lambda^*}\|_\infty < p_2(\hat{a}) < \|u_{\lambda_*}\|_\infty \quad \text{for } a = \hat{a}, \tag{1.7}$$

$$\gamma(a) < \|u_{\lambda^*}\|_\infty < p_2(a) < \|u_{\lambda_*}\|_\infty \quad \text{for } \check{a} < a < \hat{a}, \tag{1.8}$$

$$\gamma(\check{a}) < \|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty = p_2(\check{a}) \quad \text{for } a = \check{a}, \tag{1.9}$$

$$\gamma(a) < \|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty < p_2(a) \quad \text{for } a_0 < a < \check{a}, \tag{1.10}$$

$$\lim_{a \to a_0^+} \|u_{\lambda^*}\|_\infty = \lim_{a \to a_0^+} \|u_{\lambda_*}\|_\infty = \|u_{\lambda_0}\|_\infty \approx 4.896. \tag{1.11}$$

Moreover,

$$\frac{a\gamma(a)}{p_1(a)} > \frac{\|u_{\lambda_*}\|_\infty}{\|u_{\lambda^*}\|_\infty} > \frac{p_2(a)}{\|u_{\lambda_0}\|_\infty} \quad \text{for } a \geq \check{a} \quad \text{and} \quad \lim_{a \to \infty} \frac{\|u_{\lambda_*}\|_\infty}{\|u_{\lambda^*}\|_\infty} = \infty. \tag{1.12}$$

The paper is organized as follows: Section 2 contains a few lemmas needed to prove the main result. Finally, Section 3 contains the proof of the main result.

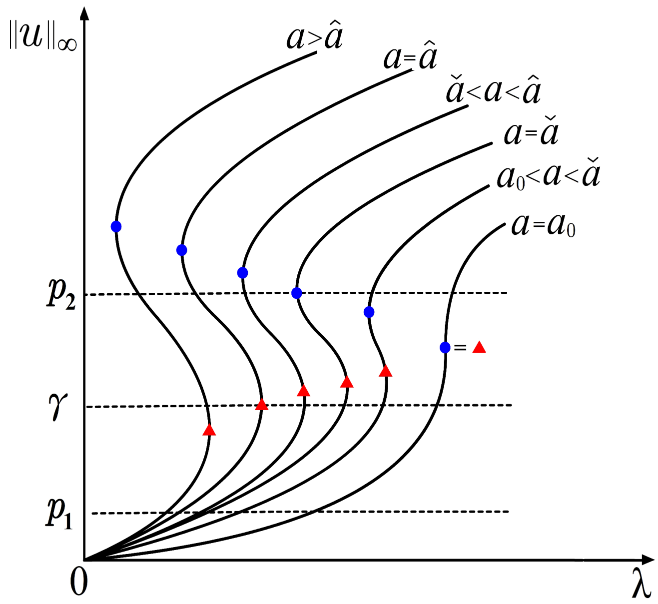

Figure 1.2: The evolution of bifurcation curves S_a with varying $a \geq a_0 \approx 4.069$. The notations • and ▲ denote the two turning points $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ and $(\lambda^*, \|u_{\lambda^*}\|_\infty)$, respectively.

2 Lemmas

To prove Theorem 1.2, we develop some new time-map techniques. The time-map formula which we apply to study (1.1) takes the form as follows:

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\alpha [F(\alpha) - F(u)]^{-1/2} du \equiv T_a(\alpha) \quad \text{for } \alpha > 0, \tag{2.1}$$

where $F(u) \equiv \int_0^u f(t)dt$, see Laetsch [12]. (Note that it can be proved that $T_a(\alpha)$ is a twice differentiable function of $\alpha > 0$ for $a > 0$, and is a differentiable function of $a > 0$ for $\alpha > 0$. The proofs are easy but tedious and hence we omit them.) So the positive solution u of (1.1) corresponds to

$$\|u\|_\infty = \alpha \quad \text{and} \quad T_a(\alpha) = \sqrt{\lambda}.$$

Thus studying the shape of bifurcation curve S_a on the $(\lambda, \|u\|_\infty)$-plane is equivalent to studying the shape of the time-map $T_a(\alpha)$ on $(0, \infty)$, cf. [6]. By (2.1) and Theorem 1.1, we note that

(i) If $a > a_0$, $T_a(\alpha)$ has exactly two critical points at $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$ where $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ are exactly two turning points of the S-shaped bifurcation curve S_a. See Figure 2.1 (i).

(ii) If $a = a_0$, $T_a(\alpha)$ has exactly one critical point at $\|u_{\lambda_0}\|_\infty$ where $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ is the unique turning point of the monotone bifurcation curve S_{a_0}. See Figure 2.1 (ii).

(iii) If $0 < a < a_0$, $T_a(\alpha)$ has no critical points on $(0, \infty)$ and is a strictly increasing function on $(0, \infty)$. See Figure 2.1 (iii).

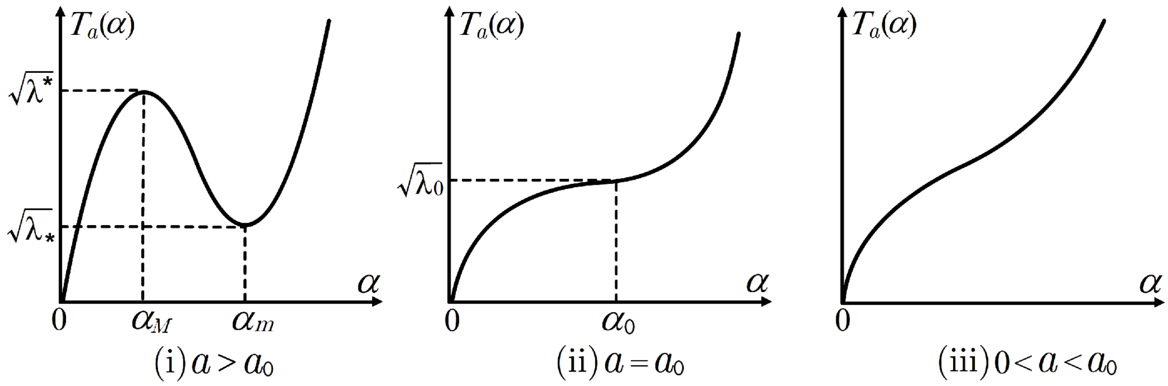

Figure 2.1: Graphs of $T_a(\alpha)$ on $(0,\infty)$ for $a > 0$. $\alpha_M = \|u_{\lambda^*}\|_\infty$, $\alpha_m = \|u_{\lambda_*}\|_\infty$ and $\alpha_0 = \|u_{\lambda_0}\|_\infty$.

For $T_a(\alpha)$ in (2.1), we compute that

$$T_a'(\alpha) = \frac{1}{2\sqrt{2}\alpha}\int_0^\alpha \frac{\theta(\alpha)-\theta(u)}{[F(\alpha)-F(u)]^{3/2}}du, \tag{2.2}$$

where

$$\theta(u) = 2F(u) - uf(u),$$

cf. [8, (3.4) and p. 230]. For the sake of convenience, we let $\gamma = \gamma(a)$, $\gamma' = \gamma'(a)$, $p_1 = p_1(a)$, $p_2 = p_2(a)$ and $p_2' = p_2'(a)$. First, we need to have the following lemma:

Lemma 2.1. *Consider* (1.1) *with* $a > 4$. *Then there exists* $\hat{a} \in [a_0, a^*)$ *such that*

$$T_a'(\gamma(a)) \begin{cases} > 0 & \text{for } 4 < a < \hat{a}, \\ = 0 & \text{for } a = \hat{a}, \\ < 0 & \text{for } \hat{a} < a \le a^* \approx 4.166. \end{cases} \tag{2.3}$$

Proof of Lemma 2.1. By (2.2), we compute that

$$\frac{\partial}{\partial a}T_a'(\gamma(a)) = \frac{1}{2\sqrt{2}\gamma^2(a)}\int_0^{\gamma(a)} \frac{N(u)}{[F(\gamma(a))-F(u)]^{5/2}}du, \tag{2.4}$$

where

$$\begin{aligned} N(u) \equiv & -[F(\gamma)-F(u)]\left\{\gamma'[\gamma f(\gamma)-uf(u)] + \gamma\int_u^\gamma \frac{s^2}{(a+s)^2}f(s)ds\right\} \\ & + \frac{3}{2}[\gamma f(\gamma)-uf(u)]\left\{\gamma'[\gamma f(\gamma)-uf(u)] + \gamma\int_u^\gamma \frac{s^2}{(a+s)^2}f(s)ds\right\} \\ & - [F(\gamma)-F(u)]\left\{\gamma' + \frac{a^2\gamma'\gamma+\gamma^3}{(a+\gamma)^2}\right\}\gamma f(\gamma) \\ & + [F(\gamma)-F(u)]\left\{\gamma' + \frac{a^2\gamma' u+\gamma u^2}{(a+u)^2}\right\}uf(u). \end{aligned}$$

By [6, Lemma 17], we have that

$$\alpha f(\alpha) - uf(u) \leq \left(1 + \frac{a}{4}\right)[F(\alpha) - F(u)] \quad \text{for } 0 \leq u \leq \alpha \text{ and } a > 4. \tag{2.5}$$

Since we compute and find that, for $0 \leq u \leq \gamma$ and $a > 4$,

$$\frac{a^2\gamma'\gamma + \gamma^3}{(a+\gamma)^2} = \gamma \quad \text{and} \quad \gamma'[\gamma f(\gamma) - uf(u)] + \gamma \int_u^\gamma \frac{s^2}{(a+s)^2} f(s)ds \geq 0,$$

and by (2.5), we obtain that

$$N(u) \leq \gamma [F(\gamma) - F(u)] L(u,a) \quad \text{for } 0 \leq u \leq \gamma \text{ and } a > 4, \tag{2.6}$$

where

$$\begin{aligned} L(u,a) \equiv &\left(\frac{3a}{8} + \frac{1}{2}\right) \int_u^\gamma \frac{s^2}{(a+s)^2} f(s)ds + \left[(a-1)\left(\frac{3a}{8} - \frac{1}{2}\right) - \gamma\right] f(\gamma) \\ &- \left[(a-1)\left(\frac{3a}{8} - \frac{1}{2}\right) - \frac{a^2(a-1)u + \gamma u^2}{[a+u]^2}\right] \frac{u}{\gamma} f(u). \end{aligned} \tag{2.7}$$

We assert that, for $4 < a \leq 4.17$,

$$L(0,a) < 0 \ \text{ and } \ \frac{\partial}{\partial u} L(u,a) \begin{cases} < 0 & \text{for } 0 \leq u < v_1, \\ = 0 & \text{for } u = v_1, \\ > 0 & \text{for } v_1 < u \leq \gamma \end{cases} \quad \text{for some } v_1 \in (0,\gamma). \tag{2.8}$$

It is easy to see that $L(\gamma, a) = 0$ by (2.7). So under (2.8), we observe that $L(u,a) < 0$ for $0 \leq u < \gamma$. So by (2.4) and (2.6), we see that $\frac{\partial}{\partial a} T_a'(\gamma(a)) < 0$ for $4 < a \leq 4.17$. It follows that $\frac{\partial}{\partial a} T_a'(\gamma(a)) < 0$ for $a_0 \leq a \leq a^*$ since $4 < a_0$ $(\approx 4.069) < a^*$ $(\approx 4.166) < 4.17$. In addition, by Theorem 1.1 (i) and (1.3), we see that

$$T_a'(\gamma(a)) \begin{cases} > 0 & \text{for } 4 < a < a_0, \\ < 0 & \text{for } a \geq a^*. \end{cases}$$

Thus there exists $\hat{a} \in [a_0, a^*)$ such that (2.3) holds. So the proof of Lemma 2.1 is complete.

Next, we divide the proof of assertion (2.8) into next Steps 1–2.

Step 1. We prove the first inequality of (2.8). We compute that

$$\int \frac{s^2}{(a+s)^2} ds = s - \frac{a^2}{a+s} - 2a \ln(a+s). \tag{2.9}$$

Since $f'(u) > 0$ for $u \geq 0$, and by (2.7) and (2.9), we compute and obtain that, for $4 < a \leq 4.17$,

$$\begin{aligned} L(0,a) &= \left(\frac{3a}{8} + \frac{1}{2}\right) \int_0^\gamma \frac{s^2}{(a+s)^2} f(s)ds + \left[(a-1)\left(\frac{3a}{8} - \frac{1}{2}\right) - \gamma\right] f(\gamma) \\ &= \left(\frac{3a}{8} + \frac{1}{2}\right) \int_0^\gamma \frac{s^2}{(a+s)^2} f(s)ds - \frac{1}{8}\left(a^2 - a - 4\right) f(\gamma) \\ &= \left(\frac{3a}{8} + \frac{1}{2}\right) \left[\int_0^2 \frac{s^2}{(a+s)^2} f(s)ds + \int_2^\gamma \frac{s^2}{(a+s)^2} f(s)ds\right] - \frac{1}{8}\left(a^2 - a - 4\right) f(\gamma) \\ &\leq \left(\frac{3a}{8} + \frac{1}{2}\right) \left[\int_0^2 \frac{s^2}{(a+s)^2} ds\right] f(2) + \left[\left(\frac{3a}{8} + \frac{1}{2}\right) \int_2^\gamma \frac{s^2}{(a+s)^2} ds - \frac{1}{8}\left(a^2 - a - 4\right)\right] f(\gamma) \\ &= \frac{1}{16(a+2)} L_1(a) < 0, \end{aligned}$$

where

$$
\begin{aligned}
L_1(a) \equiv\; & 4\,(3a+4)\left[2a+2+\left(a^2+2a\right)\ln\frac{a}{a+2}\right]\exp\left(\frac{2a}{a+2}\right)\\
& +\left[3a^4+8a^3-30a^2-84a-48+\left(12a^3+40a^2+32a\right)\ln\left(\frac{2\,(a+2)}{a^2}\right)\right]\exp\,(a-2)\\
<\; & 0 \quad \text{for } 4<a\le 4.17,
\end{aligned}
$$

see Figure 2.2. So the first inequality of (2.8) holds.

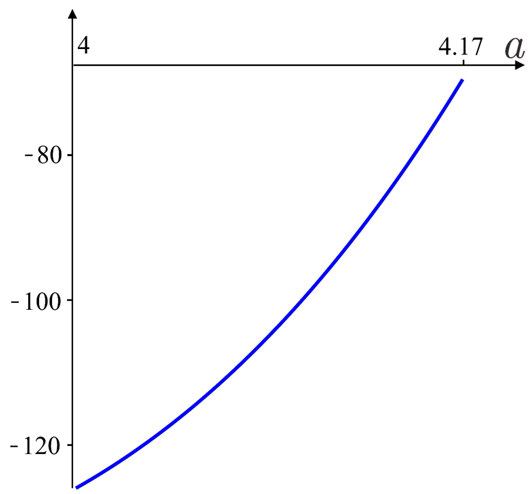

Figure 2.2: The graph of $L_1(a)$ on $[4, 4.17]$ and $L_1(4.17) \approx -69.547$.

Step2. We prove the second inequality of (2.8). We compute that

$$\frac{\partial}{\partial u}L(u,a) = \frac{f(u)}{8a\,(a-2)\,(a+u)^4}L_2(u), \tag{2.10}$$

where

$$
\begin{aligned}
L_2(u) \equiv\; & -(3a-4)\left(a^2-2\right)u^4+(-4a^4+10a^3-32a)u^3\\
& +(a^5+34a^4-4a^3-48a^2)u^2-2a^3\,(a-1)\,(3a+4)\,(a-4)\,u\\
& -2a^4\,(a-1)\,(3a-4)
\end{aligned}
$$

is a quartic polynomial of u. We compute that, for $4 < a \le 4.17$,

$$L_2(0) = -2a^4\,(a-1)\,(3a-4) < 0, \tag{2.11}$$

$$L_2(\gamma) = \frac{a^8}{16}\left\{(3a-8)\left[(4.2-a)\,(a+0.4)+\frac{5a+8}{25}\right]+8\right\} > 0, \tag{2.12}$$

$$L_2'(0) = -2a^3\,(a-1)\,(3a+4)\,(a-4) < 0, \tag{2.13}$$

$$L_2'(\gamma) = \frac{1}{2}a^6\,(3a-4)\,[(4.2-a)\,(a+0.2)+0.16] > 0, \tag{2.14}$$

$$L_2''(0) = \left[\left(2a^2-8\right)a+\left(68a^2-96\right)\right]a^2 > 0, \tag{2.15}$$

$$L_2''(\gamma) = a^3\left[(36-9a)\,a^3-10a^2+(64-40a)\right] < 0. \tag{2.16}$$

Since $L_2''(u)$ is a quadratic polynomial with a negative leading coefficient, and by (2.15) and (2.16), there exists $v_2 \in (0,\gamma)$ such that

$$L_2''(u)\begin{cases} >0 & \text{for } 0\le u<v_2,\\ =0 & \text{for } u=v_2,\\ <0 & \text{for } v_2<u\le\gamma. \end{cases}$$

So by (2.13) and (2.14), there exists $v_3 \in (0,\gamma)$ such that

$$L_2'(u)\begin{cases} <0 & \text{for } 0\le u<v_3,\\ =0 & \text{for } u=v_3,\\ >0 & \text{for } v_3<u\le\gamma. \end{cases}$$

So by (2.10)–(2.12), there exists $v_1 \in (0,\gamma)$ such that the second inequality of (2.8) holds.

The proof of Lemma 2.1 is complete. □

Lemma 2.2. *Consider* (1.1) *with* $4 < a \le 4.108$. *Then*

$$3.6\left[F(p_2)-F(u)\right]\le A(u)\le M_a\left[F(p_2)-F(u)\right] \quad \text{for } 0\le u\le p_2, \tag{2.17}$$

where

$$A(u)\equiv\frac{p_2'}{p_2}\left[p_2f(p_2)-uf(u)\right]+\int_u^{p_2}\frac{s^2}{(a+s)^2}f(s)ds,$$

$$M_a\equiv\frac{p_2'}{p_2}\left(\frac{a}{4}+1\right)+\frac{p_2^2}{(a+p_2)^2}.$$

Proof of Lemma 2.2. Let

$$U_1(u)\equiv M_a\left[F(p_2)-F(u)\right]-A(u) \quad\text{and}\quad U_2(u)\equiv A(u)-3.6\left[F(p_2)-F(u)\right].$$

To prove (2.17), it is sufficient to prove that $U_1(u)\ge 0$ and $U_2(u)\ge 0$ for $0\le u\le p_2$.

(I) We prove that $U_1(u)\ge 0$ for $0\le u\le p_2$. Clearly, we see that

$$p_2'(a)=\frac{(a-1)\sqrt{a^2-4a}+a(a-3)}{\sqrt{a^2-4a}}>0 \quad \text{for } a>4. \tag{2.18}$$

Since $u^2/(a+u)^2$ is a strictly increasing function of $u>0$ for $a>0$, and by (2.18), we compute and observe that, for $0\le u\le p_2$,

$$\begin{aligned} U_1'(u)&=\frac{d}{du}\left\{\int_u^{p_2}\left(M_a-\frac{s^2}{(a+s)^2}\right)f(s)ds-\frac{p_2'}{p_2}\left[p_2f(p_2)-uf(u)\right]\right\}\\ &=\left\{-M_a+\frac{u^2}{(a+u)^2}+\frac{p_2'}{p_2}\left[\frac{a^2u}{(a+u)^2}+1\right]\right\}f(u)\\ &=-\left\{\frac{a(a-u)^2p_2'}{4(a+u)^2p_2}+\frac{p_2^2}{(a+p_2)^2}-\frac{u^2}{(a+u)^2}\right\}f(u)<0. \end{aligned} \tag{2.19}$$

Since $U_1(p_2)=0$, and by (2.19), we see that $U_1(u)\ge 0$ for $0\le u\le p_2$. It implies that the second inequality of (2.17) holds.

(II) We prove that $U_2(u) \geq 0$ for $0 \leq u \leq p_2$. We observe that

$$U_2(u) = \frac{p_2'}{p_2}\left[p_2 f(p_2) - u f(u)\right] + \int_u^{p_2} \left(\frac{s^2}{(a+s)^2} - 3.6\right) f(s)ds.$$

First, we assert that

$$U_2(0) = p_2' f(p_2) + \int_0^{p_2} \left[\frac{s^2}{(a+s)^2} - 3.6\right] f(s)ds > 0 \quad \text{for } 4 < a \leq 4.108. \tag{2.20}$$

Indeed, by (1.4), we observe that

$$\frac{\partial}{\partial a} p_2' f(p_2) = \frac{2af(p_2)}{\left[a + \sqrt{a(a-4)}\right]\left[a(a-4)\right]^{3/2}} w_1(a) < 0 \quad \text{for } 4 < a \leq 4.108, \tag{2.21}$$

where

$$w_1(a) \equiv \sqrt{a(a-4)}\left[a(a-1)(a-4)+1\right] + a^4 - 7a^3 + 12a^2 - a.$$

See Figure 2.3(i). Clearly,

$$\begin{aligned} &\frac{d}{da}\int_0^{5.7}\left[\frac{s^2}{(a+s)^2} - 3.6\right] f(s)ds \\ &\quad = -\int_0^{5.7} \frac{s^2 f(s)}{5(a+s)^4}\left[13s^2 + (36a+10)s + 18a^2 + 10a\right] ds < 0. \end{aligned} \tag{2.22}$$

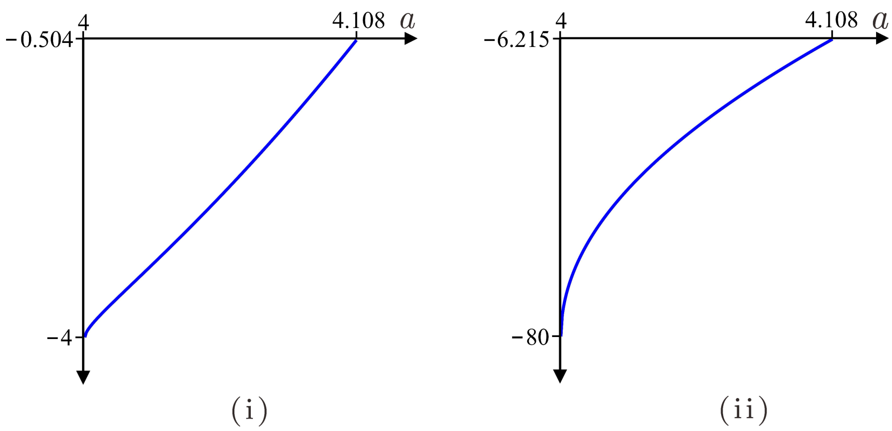

Figure 2.3: (i) The graph of $w_1(a)$ on $[4, 4.108]$. (ii) The graph of $w_2(a)$ on $[4, 4.108]$.

By (2.18), we compute that

$$p_2(a) \leq p_2(4.108) \;\; (\approx 5.697) < 5.7 \quad \text{for } 4 < a \leq 4.108. \tag{2.23}$$

So by (2.21)–(2.23), we compute and find that, for $4 < a \leq 4.108$,

$$\begin{aligned} U_2(0) &\geq p_2' f(p_2) + \int_0^{5.7}\left[\frac{s^2}{(a+s)^2} - 3.6\right] f(s)ds \\ &\geq \left\{ p_2' f(p_2) + \int_0^{5.7}\left[\frac{s^2}{(a+s)^2} - 3.6\right] f(s)ds \right\}_{a=4.108} \quad (\approx 1.174) > 0. \end{aligned}$$

Thus assertion (2.20) holds.

Secondly, we compute and obtain that, for $0 \leq u < p_2$,

$$\frac{U_2'(u)}{f(u)} = 3.6 - \frac{p_2'}{p_2}\left[\frac{a^2 u}{(a+u)^2} + 1\right] - \frac{u^2}{(a+u)^2}, \tag{2.24}$$

$$\begin{aligned}
\left(\frac{U_2'(u)}{f(u)}\right)' &= -\frac{d}{du}\left\{\frac{p_2'}{p_2}\left[\frac{a^2 u}{(a+u)^2} + 1\right] + \frac{u^2}{(a+u)^2}\right\} \\
&= \frac{a}{(a+u)^3\sqrt{a(a-4)}}\left\{\left[a - \sqrt{a(a-4)}\right]u - a^2 - a\sqrt{a(a-4)}\right\} \\
&< \frac{a}{(a+u)^3\sqrt{a(a-4)}}\left\{\left[a - \sqrt{a(a-4)}\right]p_2 - a^2 - a\sqrt{a(a-4)}\right\} \\
&= 0.
\end{aligned} \tag{2.25}$$

By (2.24), we compute and obtain that

$$\frac{U_2'(p_2)}{f(p_2)} = -2\frac{p_2'}{p_2} - \frac{p_2}{a^2} + 3.6 = \frac{w_2(a)}{10a\sqrt{a^2-4a}} < 0 \quad \text{for } 4 < a \leq 4.108, \tag{2.26}$$

where $w_2(a) \equiv \sqrt{a(a-4)}\,(31a-10) - 5a^2$. See Figure 2.3 (ii). Since $f(u) > 0$ for $u > 0$, and by (2.25) and (2.26), we see that either $U_2'(u) < 0$ for $0 < u \leq p_2$, or there exists $v_4 \in (0, p_2)$ such that

$$U_2'(u) \begin{cases} > 0 & \text{for } 0 \leq u < v_4, \\ = 0 & \text{for } u = v_4, \\ < 0 & \text{for } v_4 < u \leq p_2. \end{cases}$$

Since $U_2(p_2) = 0$, and by (2.20), we further see that $U_2(u) \geq 0$ for $0 \leq u \leq p_2$. It implies that the first inequality of (2.17) holds.

The proof of Lemma 2.2 is complete. $\square$

Lemma 2.3. *Consider* (1.1) *with $a > 4$. There exists $\check{a} \in [a_0, 4.108)$ such that*

$$T_a'(p_2(a)) \begin{cases} > 0 & \text{for } 4 < a < \check{a}, \\ = 0 & \text{for } a = \check{a}, \\ < 0 & \text{for } a > \check{a}. \end{cases} \tag{2.27}$$

Proof of Lemma 2.3. We compute that

$$\frac{\partial}{\partial a}F(p_2) = p_2' f(p_2) + \int_0^{p_2} \frac{t^2}{(a+t)^2} f(t)\,dt \tag{2.28}$$

and

$$\frac{\partial}{\partial a}p_2 f(p_2) = p_2' f(p_2) + \frac{a^2 p_2 p_2' + p_2^3}{(a+p_2)^2} f(p_2). \tag{2.29}$$

We further compute that, by (2.2), (2.28) and (2.29),

$$\begin{aligned}
\frac{\partial}{\partial a}T_a'(p_2(a)) &= \frac{\partial}{\partial a}\left\{\frac{1}{2\sqrt{2}}\int_0^1 \frac{\theta(p_2)-\theta(p_2 t)}{[F(p_2)-F(p_2 t)]^{3/2}}dt\right\} \quad (\text{let } t=\frac{u}{p_2}) \\
&= \frac{1}{2\sqrt{2}}\int_0^1 \frac{\left\{\frac{\partial}{\partial a}[\theta(p_2)-\theta(p_2 t)]\right\}[F(p_2)-F(p_2 t)]}{[F(p_2)-F(p_2 t)]^{5/2}}dt \\
&\quad - \frac{1}{2\sqrt{2}}\int_0^1 \frac{\frac{3}{2}[\theta(p_2)-\theta(p_2 t)]\frac{\partial}{\partial a}[F(p_2)-F(p_2 t)]}{[F(p_2)-F(p_2 t)]^{5/2}}dt \\
&= \frac{1}{2\sqrt{2}p_2}\int_0^{p_2} \frac{[F(p_2)-F(u)]B(u)-\frac{3}{2}[\theta(p_2)-\theta(u)]A(u)}{[F(p_2)-F(u)]^{5/2}}du, \qquad (2.30)
\end{aligned}$$

where $A(u)$ is defined in Lemma 2.2 and

$$B(u) \equiv 2A(u) - \left[p_2' + \frac{p_2\left(a^2 p_2' + p_2^2\right)}{(a+p_2)^2}\right] f(p_2) + \left[\frac{p_2'}{p_2}u + \frac{u\left(a^2 p_2' u + p_2 u^2\right)}{p_2 (a+u)^2}\right] f(u).$$

In addition, by [6, Lemma 12], we see that there exists $\bar{p}_2 \in (0, p_1)$ such that

$$\theta(p_2)-\theta(u) \begin{cases} >0 & \text{for } 0 \le u < \bar{p}_2, \\ =0 & \text{for } u = \bar{p}_2, \\ <0 & \text{for } \bar{p}_2 < u < p_2. \end{cases} \qquad (2.31)$$

So by Lemma 2.2, we observe that, for $0 \le u < \bar{p}_2$,

$$-\frac{3}{2}[\theta(p_2)-\theta(u)]A(u) \le -5.4[\theta(p_2)-\theta(u)][F(p_2)-F(u)], \qquad (2.32)$$

and, for $\bar{p}_2 \le u \le p_2$,

$$-\frac{3}{2}[\theta(p_2)-\theta(u)]A(u) \le -\frac{3}{2}M_a[\theta(p_2)-\theta(u)][F(p_2)-F(u)]. \qquad (2.33)$$

By (2.30)–(2.33), we have that

$$\begin{aligned}
\frac{\partial}{\partial a}T_a'(p_2) &\le \frac{1}{2\sqrt{2}p_2}\int_0^{p_2}\frac{U_2(u)}{[F(p_2)-F(u)]^{3/2}}du - \frac{5.4}{2\sqrt{2}p_2}\int_0^{\bar{p}_2}\frac{\theta(p_2)-\theta(u)}{[F(p_2)-F(u)]^{3/2}}du \\
&\quad - \frac{3}{4\sqrt{2}p_2}M_a\int_{\bar{p}_2}^{p_2}\frac{\theta(p_2)-\theta(u)}{[F(p_2)-F(u)]^{3/2}}du \\
&= \frac{1}{2\sqrt{2}p_2}\int_0^{\bar{p}_2}\frac{B(u)}{[F(p_2)-F(u)]^{3/2}}du + \frac{1}{2\sqrt{2}p_2}\int_{\bar{p}_2}^{p_2}\frac{C(u)}{[F(p_2)-F(u)]^{3/2}}du \\
&\quad - \frac{5.4}{2\sqrt{2}p_2}T_a'(p_2), \qquad (2.34)
\end{aligned}$$

where

$$C(u) \equiv B(u) - \left(\frac{3}{2}M_a - 5.4\right)[\theta(p_2)-\theta(u)].$$

We assert that

$$B(u) < 0 \ \text{ for } 0 < u < \bar{p}_2 \ \text{ and } \ C(u) < 0 \ \text{ for } \bar{p}_2 \le u < p_2 \text{ and } 4 < a \le 4.108. \qquad (2.35)$$

In addition, by [6, Lemma 16], there exists a positive number $\tilde{a}$ (≈ 4.107) such that $T_a'(p_2(a)) < 0$ for $a \geq \tilde{a}$. By Theorem 1.1 (iii), we see that $T_a'(p_2(a)) > 0$ for $0 < a < a_0$. It follows that there exists $\breve{a} \in [a_0, 4.108)$ such that $T_{\breve{a}}'(p_2(\breve{a})) = 0$. Furthermore, since $4 < \tilde{a} < 4.108$, and by (2.34) and (2.35), we see that

$$\frac{\partial}{\partial a} T_a'(p_2(a))\Big|_{a=\breve{a}} < 0.$$

Thus $\breve{a}$ is unique and (2.27) holds. We then divide the proof of (2.35) into next Steps 1–3.

Step 1. We prove that $1 < \bar{p}_2(a)$ for $4 < a \leq 4.108$. Let

$$\Lambda_a(u) \equiv \theta(u) - \theta(p_2) \quad \text{for } 0 \leq u \leq p_2.$$

By (2.18), we see that $1 < 4 = p_2(4) < p_2(a)$ for $a > 4$. So by (2.31), it is sufficient to prove that $\Lambda_a(1) < 0$ for $4 < a \leq 4.108$. We compute that

$$\frac{\partial}{\partial a}\Lambda_a(u) = -2\int_u^{p_2} \frac{s^2}{(a+s)^2} f(s)ds + \frac{p_2^3 f(p_2)}{(a+p_2)^2} - \frac{u^3 f(u)}{(a+u)^2}. \tag{2.36}$$

Since

$$u^2 - a^2u - a^2 < 0 \quad \text{for} \quad 0 \leq u \leq p_2 < \frac{a\left(a+\sqrt{a^2+4}\right)}{2} \quad \text{and} \quad a > 4,$$

we further compute and obtain that

$$\frac{\partial}{\partial u}\frac{\partial}{\partial a}\Lambda_a(u) = \frac{u^2 f(u)}{(a+u)^4}\left(u^2 - a^2u - a^2\right) < 0 \quad \text{for } 0 \leq u \leq p_2 \text{ and } a > 4. \tag{2.37}$$

So by (2.36) and (2.37), we have that

$$\frac{\partial}{\partial a}\Lambda_a(u) > \frac{\partial}{\partial a}\Lambda_a(u)\Big|_{u=p_2} = 0 \quad \text{for } 0 \leq u < p_2 \text{ and } a > 4. \tag{2.38}$$

By (2.38), we compute and obtain that $\Lambda_a(1) < \Lambda_{4.108}(1)$ (≈ -0.0356) < 0. Thus $1 < \bar{p}_2(a)$ for $4 < a \leq 4.108$.

Step 2. We prove that $B(u) < 0$ for $0 < u < \bar{p}_2$ and $4 < a \leq 4.108$. Clearly, $B(p_2) = 0$. By (2.38), we see that, for $a > 4$,

$$B(0) = 2\int_0^{p_2} \frac{s^2}{(a+s)^2} f(s)ds - \frac{p_2^3}{(a+p_2)^2} f(p_2) = \frac{\partial}{\partial a}\theta(p_2) = -\frac{\partial}{\partial a}\Lambda_a(0) < 0.$$

We assert that there exists $\mu_1 \in (0, p_2)$ such that

$$B'(u) \begin{cases} < 0 & \text{for } 0 \leq u < \mu_1, \\ = 0 & \text{for } u = \mu_1, \\ > 0 & \text{for } \mu_1 < u < p_2. \end{cases} \tag{2.39}$$

Thus $B(u) < 0$ for $0 \leq u < p_2$. It implies that $B(u) < 0$ for $0 < u < \bar{p}_2$.

Next, we prove assertion (2.39). We compute that

$$B'(u) = \frac{f(u)}{a\,(a+u)^4\sqrt{a^2-4a}}\bar{B}(u), \tag{2.40}$$

where

$$\bar{B}(u) \equiv a\left[-u^4 + (-a^2-4a)u^3 + (a^4-6a^2)u^2 + (a^4-4a^3)u - a^4\right] \\ + \sqrt{a^2-4a}\,(u+a)\left[(-a-1)u^3 + (a^3-3a)u^2 + (a^3-3a^2)u - a^3\right].$$

We further compute that

$$\bar{B}''(u) = -12\left[a + (a+1)\sqrt{a^2-4a}\right]u^2 + \left[-6a^3 - 24a^2 + 6a\left(a^2-a-4\right)\sqrt{a^2-4a}\right]u \\ + 2\left(a^2-6\right)a^3 + 2a^2(a+3)(a-2)\sqrt{a^2-4a}.$$

Obviously, the leading coefficient of quadratic polynomial $\bar{B}''(u)$ is negative and $\bar{B}''(0) > 0$. So there exists $\mu_2 > 0$ such that

$$\bar{B}''(u) = \begin{cases} > 0 & \text{for } 0 \le u < \mu_2, \\ = 0 & \text{for } u = \mu_2, \\ < 0 & \text{for } u > \mu_2. \end{cases} \tag{2.41}$$

We compute that, for $a > 4$,

$$\bar{B}'(0) = a^3(a-4)(a+\sqrt{a^2-4a}) > 0, \tag{2.42}$$

$$\bar{B}'(\gamma) = 2a^3\left[-2a^2 + 3a + (a-2)\sqrt{a^2-4a}\right] < 2a^3\left[-2a^2 + 3a + (a-2)\,a\right] \\ = -2a^4\,(a-1) < 0. \tag{2.43}$$

Since $\gamma(a) < p_2(a)$ for $a > 4$, and by (2.41)–(2.43), there exists $\mu_3 \in (0, p_2)$ such that

$$\bar{B}'(u) = \begin{cases} > 0 & \text{for } 0 \le u < \mu_3, \\ = 0 & \text{for } u = \mu_3, \\ < 0 & \text{for } \mu_3 < u < p_2. \end{cases} \tag{2.44}$$

We compute that $\bar{B}(0) = -a^4(a+\sqrt{a^2-4a}) < 0$ and $\bar{B}(p_2) = 0$ for $a > 4$. So by (2.40) and (2.44), assertion (2.39) holds.

Step 3. We prove that $C(u) < 0$ for $\bar{p}_2 \le u < p_2$. By Step 1, Lemma 2.2 and (2.38), we observe that, for $4 < a \le 4.108$,

$$M_a > 3.6, \quad \theta(p_2) - \theta(1) > 0, \quad \frac{a^2 p_2}{(a+p_2)^2} = 1, \tag{2.45}$$

$$2\int_1^{p_2} \frac{s^2}{(a+s)^2} f(s)ds - \frac{p_2^3}{(a+p_2)^2} f(p_2) + \frac{1}{(a+1)^2} f(1) = -\frac{\partial}{\partial a}\Lambda_a(1) < 0. \tag{2.46}$$

By (2.18), (2.45) and (2.46), we obtain that, for $4 < a \le 4.108$,

$$C(1) = 2\frac{p_2'}{p_2}\left[p_2 f(p_2) - f(1)\right] + 2\int_1^{p_2} \frac{s^2}{(a+s)^2} f(s)ds - \left[2p_2' + \frac{p_2^3}{(a+p_2)^2}\right] f(p_2) \\ + \left[\frac{p_2'}{p_2} + \frac{p_2' a^2}{p_2(a+1)^2} + \frac{1}{(a+1)^2}\right] f(1) - \left(\frac{3}{2}M_a - 5.4\right)\left[\theta(p_2) - \theta(1)\right] \\ = \frac{p_2'}{p_2}\left[\frac{a^2}{(a+1)^2} - 1\right] f(1) - \frac{3}{2}\left(M_a - 3.6\right)\left[\theta(p_2) - \theta(1)\right] - \frac{\partial}{\partial a}\Lambda_a(1) < 0. \tag{2.47}$$

We assert that there exists $\mu_4 \in (1, p_2)$ such that

$$\text{either } C'(u) > 0 \ \text{ for } 1 < u < p_2, \ \text{ or } \ C'(u) \begin{cases} < 0 & \text{for } 1 \le u < \mu_4, \\ = 0 & \text{for } u = \mu_4, \\ > 0 & \text{for } \mu_4 < u \le p_2. \end{cases} \tag{2.48}$$

By Step 1, we note that $1 < \bar{p}_2(a)$ for $4 < a \le 4.108$. Since $C(p_2) = 0$, and by (2.47) and (2.48), we see that $C(u) < 0$ for $\bar{p}_2 \le u < p_2$.

Next, we prove assertion (2.48). We compute that

$$C'(u) = \frac{f(u)}{10a(a-4)\left[a + \sqrt{a(a-4)}\right]^2 (a+u)^4} \bar{C}(u), \tag{2.49}$$

where

$$\begin{aligned} \bar{C}(u) \equiv a(a-4)\Big[&(-83a^2 + 141a + 40)u^4 + (83a^4 - 473a^3 + 444a^2 + 160a)u^3 \\ &+ (166a^5 - 680a^4 + 566a^3 + 240a^2)u^2 + (63a^6 - 353a^5 + 364a^4 + 160a^3)u \\ &- 63a^6 + 101a^5 + 40a^4\Big] \\ + \sqrt{a(a-4)}\Big[&(-83a^3 + 307a^2 + 180a)u^4 + (83a^5 - 639a^4 + 968a^3 + 720a^2)u^3 \\ &+ (166a^6 - 1012a^5 + 1162a^4 + 1080a^3)u^2 \\ &+ (63a^7 - 479a^6 + 728a^5 + 720a^4)u - 63a^7 + 227a^6 + 180a^5\Big]. \end{aligned}$$

We further compute that $\bar{C}''(u) = \psi_2(a)u^2 + \psi_1(a)u + \psi_0(a)$ where

$$\begin{aligned} \psi_2(a) &\equiv -12a(a-4)\left(83a^2 - 141a - 40\right) - 12a\sqrt{a(a-4)}\left(83a^2 - 307a - 180\right), \\ \psi_1(a) &\equiv 6a^2(a-4)\left(83a^3 - 473a^2 + 444a + 160\right) \\ &\quad - 6a^2\sqrt{a(a-4)}(-83a^3 + 639a^2 - 968a - 720), \\ \psi_0(a) &\equiv 4a^3(a-4)\left(83a^3 - 340a^2 + 283a + 120\right) \\ &\quad + 4a^3\sqrt{a(a-4)}(83a^3 - 506a^2 + 581a + 540). \end{aligned}$$

Since $83a^2 - 307a - 180 < 0$ for $4 < a \le 4.108$, we observe that, for $4 < a \le 4.108$,

$$\begin{aligned} \psi_2(a) &\le -12a(a-4)\left(83a^2 - 141a - 40\right) - 12a^2(a-4)\left(83a^2 - 307a - 180\right) \\ &= -12a(a-4)\left(83a^3 - 224a^2 - 321a - 40\right) < 0. \end{aligned}$$

It implies that the quadratic polynomial $\bar{C}''(u)$ of u has a negative leading coefficient. Similarly, we observe that $\bar{C}''(0) = \psi_0(a) > 0$ for $4 < a \le 4.108$. Then there exists $\mu_5 > 0$ such that

$$\bar{C}''(u) \begin{cases} > 0 & \text{for } 0 \le u < \mu_5, \\ = 0 & \text{for } u = \mu_5, \\ < 0 & \text{for } u > \mu_5. \end{cases} \tag{2.50}$$

From Figure 2.4, we further see that, for $4 < a \le 4.108$,

$$\begin{aligned}\bar{C}'(1) = {} & a(a-4)(63a^6 - 21a^5 - 747a^4 - 127a^3 + 1480a^2 + 1044a \\ & + 160) + a\sqrt{a\,(a-4)}(63a^6 - 147a^5 - 1047a^4 + 1127a^3 \\ & + 4732a^2 + 3388a + 720) > 0,\end{aligned} \tag{2.51}$$

$$\begin{aligned}\bar{C}'(p_2(a)) = {} & -a^6(a-4)(83a^3 - 473a^2 + 539a + 140) \\ & - a^5\sqrt{a\,(a-4)}\left(83a^4 - 639a^3 + 1319a^2 - 324a - 80\right) < 0.\end{aligned} \tag{2.52}$$

By (2.50)–(2.52), for $4 < a \le 4.108$, there exists $\mu_6 \in (1, p_2)$ such that

$$\bar{C}'(u) \begin{cases} > 0 & \text{for } 1 \le u < \mu_6, \\ = 0 & \text{for } u = \mu_6, \\ < 0 & \text{for } \mu_6 < u \le p_2. \end{cases} \tag{2.53}$$

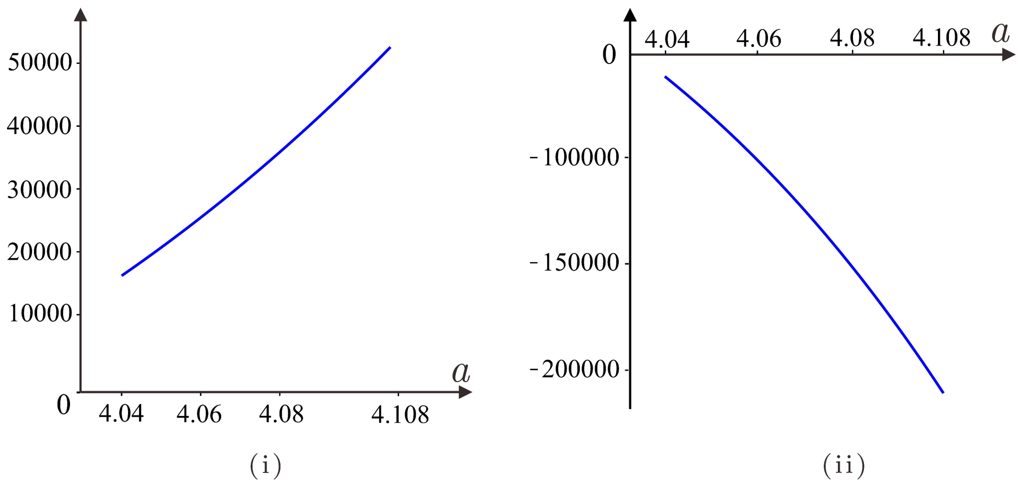

Figure 2.4: (i) The graph of $\bar{C}'(1)$ on $[4, 4.108]$. (ii) The graph of $\bar{C}'(p_2(a))$ on $[4, 4.108]$.

We compute that $\bar{C}(p_2) = 0$. So by (2.53), we see that either $\bar{\Psi}_3(u) > 0$ for $1 < u < p_2$, or

$$\bar{C}(u) \begin{cases} < 0 & \text{for } 1 \le u < \eta_6, \\ = 0 & \text{for } u = \eta_6, \\ > 0 & \text{for } \eta_6 < u \le p_2 \end{cases} \quad \text{for some } \eta_6 \in (1, p_2).$$

So by (2.49), (2.48) holds.

The proof of Lemma 2.3 is complete. □

By numerical simulations, we compute and find that

(i) $T'_{4.075}(4.8)\ \left(\approx -3.461 \times 10^{-4}\right) < 0$,

(ii) $T'_{4.075}(p_2(4.075))\ \left(\approx 6.596 \times 10^{-5}\right) > 0$,

(iii) $T'_{4.084}(\gamma(4.084))$ $(\approx 3.351 \times 10^{-4}) > 0$,

(iv) $T'_{4.084}(p_2(4.084))$ $(\approx -2.474 \times 10^{-4}) < 0$.

In fact, these inequalities can be proved by analytic techniques, see the next lemma. These results as stated in next lemma are needed in the proof of Theorem 1.2.

Lemma 2.4. *Consider* (1.1). *The following assertions (i)–(iv) hold.*

(i) $T'_{4.075}(4.8) < 0$.

(ii) $T'_{4.075}(p_2(4.075)) = T'_{4.075}\left(\frac{13529}{3200} + \frac{163}{3200}\sqrt{489}\right) > 0$.

(iii) $T'_{4.084}(\gamma(4.084)) = T'_{4.084}\left(\frac{531941}{125000}\right) > 0$.

(iv) $T'_{4.084}(p_2(4.084)) = T'_{4.084}\left(\frac{531941}{125000} + \frac{1021}{125000}\sqrt{21441}\right) < 0$.

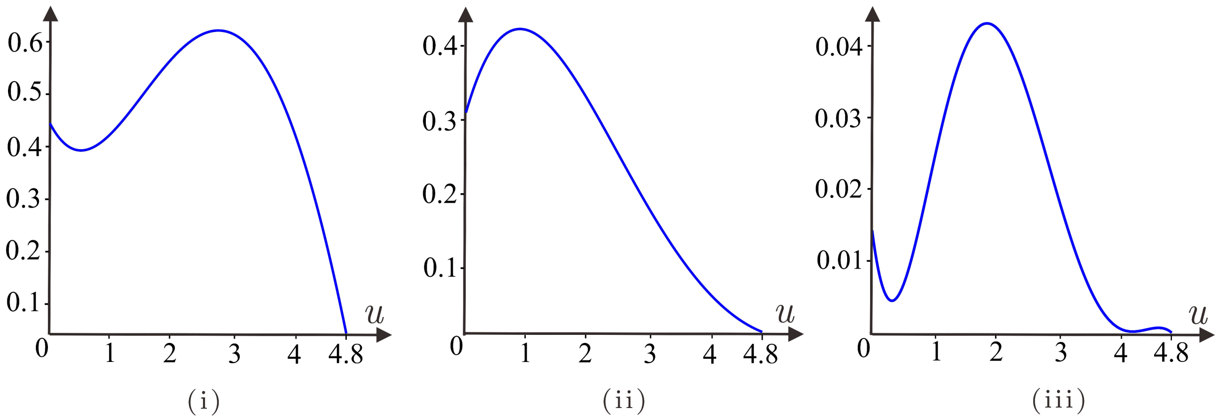

Figure 2.5: (i) The graph of $F(4.8) - F(u) - X_1(u)$ on $[0, 4.8]$. (ii) The graph of $X_2(u) - F(4.8) + F(u)$ on $[0, 4.8]$. (iii) The graph of $4.8f(4.8) - uf(u) - X_3(u)\,[F(4.8) - F(u)]$ on $[0, 4.8]$. Note that $a = 4.075$.

Proof of Lemma 2.4. The proofs of assertions (i)–(iv) are similar. So we only prove assertion (i) while the proofs of assertions (ii)–(iv) are omitted. Let

$$X_1(u) \equiv -\frac{1}{5}\left(u - \frac{24}{5}\right)(4u + 23), \qquad X_2(u) \equiv -\frac{1}{10}\left(u - \frac{24}{5}\right)(9u + 48),$$

$$X_3(u) \equiv -\frac{29}{5000}\left(u - \frac{383}{100}\right)^2 + \frac{10083}{5000}.$$

Assume that $a = 4.075$. By Figure 2.5, we obtain that

$$X_1(u) < F(4.8) - F(u) < X_2(u) \quad \text{for } 0 \leq u \leq 4.8, \tag{2.54}$$

$$X_3(u)\,[F(4.8) - F(u)] \leq 4.8f(4.8) - uf(u) \quad \text{for } 0 \leq u \leq 4.8. \tag{2.55}$$

The proofs of (2.54) and (2.55) are trivial but rather lengthy, and hence we put them in [7]. Clearly, the quartic polynomials $X_1(u) > 0$ and $X_2(u) > 0$ for $0 \leq u < 4.8$.

We further see that there exists

$$\varsigma \equiv \frac{383}{100} - \frac{1}{29}\sqrt{2407} \approx 2.138$$

such that $0 < X_3(u) < 2$ for $0 \leq u < \varsigma$, $X_3(\varsigma) = 2$ and $X_3(u) > 2$ for $\varsigma < u \leq 4.8$. So by (2.54) and (2.55), we observe that

$$\begin{aligned}
T'_{4.075}(4.8) &= \frac{5}{48\sqrt{2}} \int_0^{4.8} \frac{2\left[F(4.8) - F(u)\right] - 4.8f(4.8) + uf(u)}{\left[F(4.8) - F(u)\right]^{3/2}} du \\
&\leq \frac{5}{48\sqrt{2}} \int_0^{4.8} \frac{2 - X_3(u)}{\sqrt{F(4.8) - F(u)}} du \\
&= \frac{5}{48\sqrt{2}} \left[\int_0^{\varsigma} \frac{2 - X_3(u)}{\sqrt{F(4.8) - F(u)}} du + \int_{\varsigma}^{4.8} \frac{2 - X_3(u)}{\sqrt{F(4.8) - F(u)}} du \right] \\
&\leq \frac{5}{48\sqrt{2}} \left[\int_0^{\varsigma} \frac{2 - X_3(u)}{\sqrt{X_1(u)}} du + \int_{\varsigma}^{4.8} \frac{2 - X_3(u)}{\sqrt{X_2(u)}} du \right].
\end{aligned}$$

We compute that

$$\begin{aligned}
\int_0^{\varsigma} \frac{2 - X_3(u)}{\sqrt{X_1(u)}} du = &\left[\left(-\frac{29u}{40000} + \frac{97121}{8 \times 10^6} \right) \sqrt{-20u^2 - 19u + 552} \right. \\
&\left. + \frac{68634343\sqrt{5}}{8 \times 10^8} \arcsin\left(\frac{40}{211}u + \frac{19}{211} \right) \right]_0^{\varsigma} \\
\approx &\ 0.01391,
\end{aligned}$$

$$\begin{aligned}
\int_{\varsigma}^{4.8} \frac{2 - X_3(u)}{\sqrt{X_2(u)}} du = &\left[\left(-\frac{29u}{90000} + \frac{11687}{2250000} \right) \sqrt{-90u^2 - 48u + 2304} \right. \\
&\left. + \frac{23277863\sqrt{10}}{45 \times 10^7} \arcsin\left(\frac{15}{76}u + \frac{1}{19} \right) \right]_{\varsigma}^{4.8} \\
\approx & -0.01455.
\end{aligned}$$

Thus we obtain that

$$T'_{4.075}(4.8) \leq \frac{5}{48\sqrt{2}} \left[\int_0^{\varsigma} \frac{2 - X_3(u)}{\sqrt{X_1(u)}} du + \int_{\varsigma}^{4.8} \frac{2 - X_3(u)}{\sqrt{X_2(u)}} du \right] \ \left(\approx -4.7 \times 10^{-5}\right) < 0.$$

The proof of Lemma 2.4 is complete. □

3 Proof of the main result

Since $\lim_{a\to\infty} p_1(a) = \lim_{a\to\infty} \frac{a(a-2) - a\sqrt{a(a-4)}}{2} = 1$ and

$$p_1'(a) = \frac{(a-1)\sqrt{a^2 - 4a} - a(a-3)}{\sqrt{a^2 - 4a}} < 0 \quad \text{for } a > 4,$$

we obtain that $p_1(a) > 1$ for $a > 4$. Assume that $a > a_0$. By Theorem 1.1, we see that $T_a(\alpha)$ has exactly two critical points, a local maximum at some $\alpha_M(a) = \|u_{\lambda^*}\|_\infty$ and a local minimum at some $\alpha_m(a) = \|u_{\lambda_*}\|_\infty$ $(> \alpha_M(a))$, see Figure 2.1. By [6, Lemma 25], we have that

$$\alpha_M(a) < \lim_{a \to a_0^+} \alpha_M(a) = \lim_{a \to a_0^+} \alpha_m(a) = \|u_{\lambda_0}\|_\infty < \alpha_m(a). \tag{3.1}$$

Thus (1.11) holds immediately. By [6, Lemma 12], we see that $\theta(p_1) - \theta(u) > 0$ for $0 \le u < p_2$ and $a > 4$. So by (2.2), we further see that $T_a'(p_1) > 0$ for $a > 4$. Since $a_0 > 4$, we see that $p_1(a) < \alpha_M(a)$ for $a > a_0$. In addition, since $4.8 < p_2(4.075)$ (≈ 5.354), and by Lemmas 2.1, 2.3 and 2.4, we see that

$$a_0 < 4.075 < \check{a} < 4.084 < \hat{a}, \tag{3.2}$$

$$(1 <)\ \ p_1(a) < \alpha_M(a) < \gamma(a) < p_2(a) < \alpha_m(a) \quad \text{ for } a > \hat{a}. \tag{3.3}$$

By Lemma 2.1 and (3.2), it is easy to see that $\gamma(\hat{a}) = \alpha_m(\hat{a})$ or $\gamma(\hat{a}) = \alpha_M(\hat{a})$. Suppose to the contrary that $\alpha_M(\hat{a}) < \alpha_m(\hat{a}) = \gamma(\hat{a})$. By [6, Lemma 25 (i)], we see that $\gamma(a)$ and $\alpha_M(a)$ are continuous functions of $a > a_0$. So by Lemma 2.1, we observe that $\alpha_M(a) < \alpha_m(a) < \gamma(a)$ for $a_0 < a < \hat{a}$. It implies that $T_a(\alpha)$ has two critical points on $(0, \gamma)$, which is a contradiction by [12, Lemma 3.2]. Thus $\gamma(\hat{a}) = \alpha_M(\hat{a})$. Then since $\gamma'(a) = a - 1 > 0$ for $a > 4$, and by [6, Lemma 25 (i)], we see that $\gamma(a)$ and $\alpha_M(a)$ are strictly increasing and strictly decreasing on (a_0, ∞), respectively. So we obtain that

$$\begin{cases} \gamma(a) = \alpha_M(a) & \text{for } a = \hat{a}, \\ \gamma(a) < \alpha_M(a) & \text{for } a_0 < a < \hat{a}. \end{cases} \tag{3.4}$$

By Lemma 2.3, we have that $\alpha_M(a) < p_2(a) < \alpha_m(a)$ for $a > \check{a}$. So by (3.2) and (3.4),

$$\begin{cases} \gamma(\hat{a}) = \alpha_M(\hat{a}) < p_2(\hat{a}) < \alpha_m(\hat{a}) & \text{for } a = \hat{a}, \\ \gamma(a) < \alpha_M(a) < p_2(a) < \alpha_m(a) & \text{for } \check{a} < a < \hat{a}. \end{cases} \tag{3.5}$$

By Lemma 2.3 and (3.2), it is easy to see that $p_2(\check{a}) = \alpha_M(\check{a})$ or $p_2(\check{a}) = \alpha_m(\check{a})$. Suppose to the contrary that $p_2(\check{a}) = \alpha_M(\check{a}) < \alpha_m(\check{a})$. Since $p_2(a)$ and $\alpha_M(a)$ are strictly increasing and strictly decreasing on (a_0, ∞) respectively, and by (2.18) and (3.2), we obtain that

$$4.8 < (5.35 \approx)\ \ p_2(4.075) < p_2(\check{a}) = \alpha_M(\check{a}) < \alpha_M(4.075).$$

It follows that $T_{4.075}'(4.8) > 0$, which is a contradiction by Lemma 2.4(i). Thus $\alpha_M(\check{a}) < \alpha_m(\check{a}) = p_2(\check{a})$. By Lemma 2.3 and continuity of $\alpha_M(a)$ and $p_2(a)$ on (a_0, ∞), we find that $\alpha_M(a) < \alpha_m(a) < p_2(a)$ for $a \in (a_0, \check{a})$. Thus we have that

$$\begin{cases} \gamma(\check{a}) < \alpha_M(\check{a}) < \alpha_m(\check{a}) = p_2(\check{a}) & \text{for } a = \check{a}, \\ \gamma(a) < \alpha_M(a) < \alpha_m(a) < p_2(a) & \text{for } a_0 < a < \check{a}. \end{cases} \tag{3.6}$$

By (3.3), (3.5) and (3.6), inequalities (1.6)–(1.10) hold.

Finally, we prove (1.12). We compute and observe that

$$\theta'(u) = \frac{t^2 - (a^2 - 2a)\,t + a^2}{(a+t)^2} f(t) \begin{cases} > 0 & \text{for } u \in (0, p_1) \cup (p_2, \infty)\,, \\ = 0 & \text{for } u \in \{p_1, p_2\}\,, \\ < 0 & \text{for } u \in (p_1, p_2) \end{cases} \qquad \text{for } a > 4. \tag{3.7}$$

Since

$$a\gamma(a) - p_2(a) = \frac{a(a-1)(a-2) - a\sqrt{a^2-4a}}{2} > \frac{a(a-1)(a-2) - a^2}{2}$$
$$= \frac{a\left(a^2-4a+2\right)}{2} > 0 \quad \text{for } a \geq \check{a} > 4,$$

we see that $p_1(a) < p_2(a) < a\gamma(a)$ for $a \geq \check{a}$. Since $f'(u) > 0$ for $u > 0$, and by (3.7), we compute and observe that

$$\begin{aligned}
\theta(a\gamma) - \theta(p_1) &= \int_{p_1}^{a\gamma} \theta'(t)dt = \int_{p_1}^{p_2} \theta'(t)dt + \int_{p_2}^{a\gamma} \theta'(t)dt \\
&\geq f(p_2)\left[\int_{p_1}^{p_2} \frac{t^2 - \left(a^2-2a\right)t + a^2}{(a+t)^2}dt + \int_{p_2}^{a\gamma} \frac{t^2 - \left(a^2-2a\right)t + a^2}{(a+t)^2}dt\right] \\
&= f(p_2)\int_{p_1}^{a\gamma} \frac{t^2 - \left(a^2-2a\right)t + a^2}{(a+t)^2}dt = f(p_2)\left[t - \frac{a^3}{a+t} - a^2\ln(a+t)\right]_{p_1}^{a\gamma} \\
&= \frac{a}{2\left(a^2-2a+2\right)\left[a - \sqrt{a\left(a-4\right)}\right]}K(a),
\end{aligned} \tag{3.8}$$

where

$$K(a) \equiv a(a^4 - 6a^3 + 20a^2 - 32a + 20) - \sqrt{a(a-4)}(a^4 - 6a^3 + 12a^2 - 16a + 4)$$
$$- 2a(a^2-2a+2)\left[a - \sqrt{a(a-4)}\right]\ln\left(\frac{a^2-2a+2}{a - \sqrt{a(a-4)}}\right).$$

From Figure 3.1, we observe that $K(a)$ is a strictly increasing and positive function of $a \geq 4.06$.

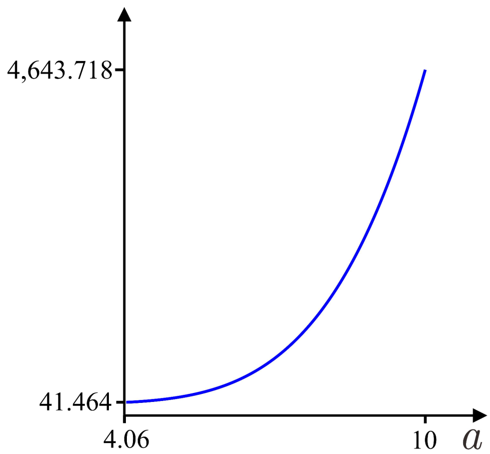

Figure 3.1: The graph of $K(a)$ for $a \geq 4.06$.

Since $\check{a} > a_0$ (≈ 4.069) > 4.06, and by (3.8), we have that $\theta(a\gamma) - \theta(p_1) > 0$ for $a \geq \check{a}$. Since $\theta(0) = 0$, and by (3.7), we observe that $\theta(\alpha) > \theta(u)$ for $0 < u < a\gamma(a)$, $\alpha \geq a\gamma(a)$ and $a \geq \check{a}$. So by (2.2), we obtain that $T_a'(\alpha) > 0$ for $\alpha \geq a\gamma(a)$ and $a \geq \check{a}$. It follows that $\alpha_m(a) < a\gamma(a)$ for $a \geq \check{a}$. So by (1.6)–(1.9) and (3.1), we see that

$$\frac{a\gamma(a)}{p_1(a)} > \frac{\alpha_m(a)}{\alpha_M(a)} = \frac{\|u_{\lambda_*}\|_\infty}{\|u_{\lambda^*}\|_\infty} > \frac{p_2(a)}{\|u_{\lambda_0}\|_\infty} \quad \text{for } a \geq \check{a},$$

$$\lim_{a\to\infty}\frac{\|u_{\lambda_*}\|_\infty}{\|u_{\lambda^*}\|_\infty}=\lim_{a\to\infty}\frac{\alpha_m(a)}{\alpha_M(a)}>\lim_{a\to\infty}\frac{p_2(a)}{\|u_{\lambda_0}\|_\infty}=\frac{1}{\|u_{\lambda_0}\|_\infty}\lim_{a\to\infty}\frac{a\,(a-2)+a\sqrt{a(a-4)}}{2}=\infty.$$

Thus (1.12) holds.

The proof of Lemma 1.2 is complete. □

Remark 3.1. By numerical simulations, we find that $\hat{a}\approx 4.088$ and $\check{a}\approx 4.077$.

Acknowledgements

Most of the computation in this paper has been checked by using the symbolic manipulator *Maple 16*. This work is partially supported by the National Science Council of the Republic of China under grant No. NSC 101-2115-M-007-001-MY2.

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